

Stability of Fuzzy $S^2 \times S^2 \times S^2$ in IIB Type Matrix Models

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Abstract

We study the stability of fuzzy $S^2 \times S^2 \times S^2$ backgrounds in three different IIB type matrix models with respect to the change of the spins of each S^2 at the two loop level. We find that $S^2 \times S^2 \times S^2$ background is metastable and the effective action favors a single large S^2 in comparison to the remaining $S^2 \times S^2$ in the models with Myers term. On the other hand, we find that a large $S^2 \times S^2$ in comparison to the remaining S^2 is favored in IIB matrix model itself. We further study the stability of fuzzy $S^2 \times S^2$ background in detail in IIB matrix model with respect to the scale factors of each S^2 as well. In this case, we find unstable directions which lower the effective action away from the most symmetric fuzzy $S^2 \times S^2$ background.

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1 Introduction

Why the spacetime is four dimensional is an interesting question in our quest to understand the origin of universe. Nowadays string theory is a leading candidate of quantum gravity, and IIB matrix model is a candidate of a non-perturbative formulation of string theory [1, 2]. Matrix models may be useful to answer this question if they can consistently describe the dynamics of gauge fields and quantum gravity as expected.

In IIB matrix model, the spacetime may be represented by a particular configuration of matrices. When we are given a matrix configuration as a background field, we can examine its stability by investigating the behavior of the effective action under the change of some parameters of the background. In this framework, the stabilities of fuzzy S^2 , $S^2 \times S^2$ [7, 8] and $T^2 \times T^2$ [10] have been studied. This paper extends these investigations into a simple 6 dimensional manifold.

In the previous investigations, the large N scaling behavior of the NC gauge theory on these manifolds has been clarified [7]. In supersymmetric models, the effective action scales as N^2 , N and $N^{\frac{4}{3}}$ in 2, 4 and 6 dimensional manifolds respectively ¹. It always scales as N^2 and the one loop approximation is exact in bosonic models [9]. A major purpose of this paper is to explicitly verify the predicted scaling behavior of the effective action ($N^{\frac{4}{3}}$) at the two loop level for a simple 6d fuzzy manifold.

In Gaussian approximations, it has been found that 4 dimensional spacetime tends to minimize the effective action [11, 12]. The advantage of our approach is that the effective actions on homogeneous spaces are much lower than those in the Gaussian approximations which are $O(N^2)$. It is because supersymmetry is broken only softly on homogeneous spaces. Although IIB matrix model is supersymmetric, supersymmetry cannot be respected on compact homogeneous spacetime. It is precisely why we obtain nonvanishing effective action on these manifolds. It has been a great challenge to explain 4 dimensionality in string theory since the vacuum degenerates as long as it is supersymmetric. In nonperturbative formulation of string theory, the extension of (Euclidean) spacetime may be inevitably finite. Such a view is consistent with finite entropy of de Sitter spacetime in which we are likely to live in. IIB matrix model suggests that the vacuum degeneracy may be lifted due to finite extension of spacetime.

¹We subtract the universal gauge volume of $SU(N)/Z_N$ from the matrix model effective action.

Fuzzy S^2 can be embedded in Hermitian matrices as

$$A_i = f j_i, \quad (1.1)$$

where j_i are the generators of $SU(2)$ with spin l and f denotes the scale factor of this background. The spin l determines the size of S^2 in our group theoretic construction of $S^2 = SU(2)/U(1)$ and we can also introduce the scale factor f to vary the overall size of S^2 . Thus the parameters of S^2 in matrix models are a spin l and a scale factor f for each S^2 .

After setting up our calculation procedure in section 2, we first investigate the stabilities of $S^2 \times S^2 \times S^2$ in three different matrix models with respect to the variation of the spins while assuming the identical scale factor f for all S^2 's. In section 3 and 4 we find that $S^2 \times S^2 \times S^2$ background is metastable and the effective action favors a single large S^2 in comparison to the remaining $S^2 \times S^2$ in the matrix models with Myers term. On the other hand, we find that a large $S^2 \times S^2$ in comparison to the remaining S^2 is favored in IIB matrix model itself in section 5. These findings are consistent with previous studies [7, 8]. We subsequently investigate $S^2 \times S^2$ and $S^2 \times S^2 \times S^2$ in IIB matrix model with respect to the variations of the spins and scale factors in section 6 and 7. In this case, we find unstable directions which lower the effective action away from the most symmetric fuzzy $S^2 \times S^2$ background as suggested by [10]. We conclude in section 8 with discussions.

2 IIB type matrix models on fuzzy $S^2 \times S^2 \times S^2$

Since our motivation is to explain the 4 dimensionality of spacetime in superstring theory, it is natural to investigate the stability of fuzzy $S^2 \times S^2 \times S^2$ in IIB matrix model [1]

$$S_{IIB} = -\frac{1}{4} \text{Tr} [A_\mu, A_\nu]^2 - \frac{1}{2} \text{Tr} \bar{\psi} \Gamma_\mu [A_\mu, \psi], \quad (2.1)$$

where A_μ and ψ are $N \times N$ Hermitian matrices. We also investigate a deformed model with Myers term [3, 4, 5, 6]

$$S_{Myers} = \frac{i}{3} f_{\mu\nu\rho} \text{Tr} [A_\mu, A_\nu] A_\rho. \quad (2.2)$$

It is because fuzzy S^2 , $S^2 \times S^2$ and $S^2 \times S^2 \times S^2$ become classical solutions after such a deformation since the equations of motion are

$$\begin{aligned} [A_\mu, [A_\mu, A_\nu]] + \{\bar{\psi}, \Gamma_\nu \psi\} &= \begin{cases} -i f_{\mu\nu\rho} [A_\mu, A_\rho] & \text{(with Myers term)} \\ 0 & \text{(without Myers term)} \end{cases}, \\ \Gamma_\mu [A_\mu, \psi] &= 0. \end{aligned} \quad (2.3)$$

This modification does not alter the convergence properties of IIB matrix model [13] while it breaks supersymmetry softly.

After adopting fuzzy $S^2 \times S^2 \times S^2$ as a background field, we separate A_μ and ψ into background fields p_μ , χ and quantum fluctuations a_μ , φ .

$$\begin{aligned} A_\mu &= p_\mu + a_\mu, \\ \psi &= \chi + \varphi. \end{aligned} \tag{2.4}$$

Since fuzzy S^2 can be realized as in (1.1), we take the following p_μ and χ to represent fuzzy $S^2 \times S^2 \times S^2$ as

$$\begin{aligned} p_\mu &= f(\bar{j}_\mu \otimes 1 \otimes 1) \otimes 1_n & (\mu = 1, 2, 3), \\ p_\mu &= f(1 \otimes \hat{j}_\mu \otimes 1) \otimes 1_n & (\mu = 4, 5, 6), \\ p_\mu &= f(1 \otimes 1 \otimes \tilde{j}_\mu) \otimes 1_n & (\mu = 7, 8, 9), \\ p_0 &= 0, \\ \chi &= 0. \end{aligned} \tag{2.5}$$

where \bar{j}_μ , \hat{j}_μ and \tilde{j}_μ are the angular momentum operators with spin l_1 , l_2 and l_3 respectively. 1_n represents the $n \times n$ identity corresponding to n coincident fuzzy $S^2 \times S^2 \times S^2$. This background is a classical solution when Myers term is added while it becomes a quantum solution in IIB matrix model at the two loop level when the effective action is stationary with respect to f .

p_μ 's satisfy the $SU(2)$ algebras as follows

$$\begin{aligned} [p_\mu, p_\nu] &= if_{\mu\nu\rho} p_\rho, \\ f_{\mu\nu\rho} &= \begin{cases} f\epsilon_{\mu\nu\rho} & (\mu, \nu, \rho) \in (1, 2, 3) \\ f\epsilon_{\mu\nu\rho} & (\mu, \nu, \rho) \in (4, 5, 6) \\ f\epsilon_{\mu\nu\rho} & (\mu, \nu, \rho) \in (7, 8, 9) \\ 0 & (\text{others}) \end{cases}. \end{aligned}$$

We can construct the Casimir operators in the respective representations as

$$\begin{aligned} \sum_{\mu=1,2,3} (p_\mu)^2 &= f^2 l_1(l_1 + 1), \\ \sum_{\mu=4,5,6} (p_\mu)^2 &= f^2 l_2(l_2 + 1), \\ \sum_{\mu=7,8,9} (p_\mu)^2 &= f^2 l_3(l_3 + 1). \end{aligned} \tag{2.6}$$

Since the right hand side of the above equations are the squared radii, the spin and scale factor f determine the size of each S^2 . The dimension of the matrices is

$$N = n(2l_1 + 1)(2l_2 + 1)(2l_3 + 1). \quad (2.7)$$

As we can see, this background represents a simple six dimensional spacetime. The choice of S^2 facilitates us to carry out detailed analysis of the effective action since the representations of $SU(2)$ are well known. By varying the representations (spins) of respective S^2 , we can explore the whole range of the manifolds from the 2 dimensional limit with a single large S^2 to the 6 dimensional limit with a large $S^2 \times S^2 \times S^2$ with the identical radii. We can thus explore which dimensionality between 2 and 6 is favored by the effective action in this class of backgrounds.

The effective action can be evaluated in a background gauge method by substituting (2.4) for (2.1) and (2.2). We introduce a gauge fixing term S_{gf} and a ghost term S_{gh} for gauge fixing. The gauge fixed action is

$$\begin{aligned} S'_{IIB} &\equiv S_{IIB} + S_{gh} + S_{gf} \\ &= -\frac{1}{4}Tr[p_\mu, p_\nu]^2 - Tra_\rho([p_\mu, [p_\rho, p_\mu]]) \\ &\quad + \frac{1}{2}Tra_\mu(\delta^{\mu\nu}P^2 + 2if_{\mu\nu\rho}P_\rho)a_\nu - \frac{1}{2}Tr\bar{\varphi}\Gamma^\mu P_\mu\varphi + TrbP^2c \\ &\quad - TrP_\mu a_\nu[a_\mu, a_\nu] - \frac{1}{4}Tr[a_\mu, a_\nu]^2 \\ &\quad - \frac{1}{2}Tr\bar{\varphi}\Gamma_\mu[a_\mu, \varphi] + TrbP_\mu[a_\mu, c], \end{aligned} \quad (2.8)$$

$$\begin{aligned} S_{Myers} &= \frac{i}{3}f_{\mu\nu\rho}Tr[p_\mu, p_\nu]p_\rho + if_{\mu\nu\rho}Tr[p_\mu, p_\nu]a_\rho \\ &\quad - if_{\mu\nu\rho}Tra_\mu P_\rho a_\nu + \frac{i}{3}f_{\mu\nu\rho}Tr[a_\mu, a_\nu]a_\rho. \end{aligned} \quad (2.9)$$

where $P_\mu X = [p_\mu, X]$, and

$$\begin{aligned} S_{gh} &= TrbP_\mu[p_\mu + a_\mu, c], \\ S_{gf} &= -\frac{1}{2}Tr(P_\mu a_\mu)^2. \end{aligned} \quad (2.10)$$

3 Bosonic matrix model with Myers term

In this section, we investigate a bosonic matrix model with Myers term whose action is

$$S = -\frac{1}{4}Tr[A_\mu, A_\nu]^2 + \frac{i}{3}f_{\mu\nu\rho}Tr[A_\mu, A_\nu]A_\rho, \quad (3.1)$$

where $\mu = 0, 1, \dots, 9$ in correspondence to IIB matrix model. We certainly require 9 matrices to construct fuzzy $S^2 \times S^2 \times S^2$. Although we are eventually interested in IIB matrix model, bosonic models are simple enough to admit exact solutions in the large N limit. Our prediction can be verified by comparing with numerical investigations [9]. For this purpose, it is useful to generalize the classical solutions as follows:

$$\begin{aligned}
p_\mu &= \beta (\bar{j}_\mu \otimes 1 \otimes 1) \otimes 1_n & (\mu = 1, 2, 3), \\
p_\mu &= \beta (1 \otimes \hat{j}_\mu \otimes 1) \otimes 1_n & (\mu = 4, 5, 6), \\
p_\mu &= \beta (1 \otimes 1 \otimes \tilde{j}_\mu) \otimes 1_n & (\mu = 7, 8, 9), \\
p_0 &= 0, \\
\chi &= 0.
\end{aligned} \tag{3.2}$$

The shift of β away from the classical value of f is required to take account of quantum effects. Since the tree and the one loop contributions dominate in bosonic models in the large N limit as we shall see, this approach enables us an exact investigation within this class of the backgrounds.

The tree level effective action is

$$\begin{aligned}
\Gamma_{tree} &= -\frac{1}{4} Tr[p_\mu, p_\nu]^2 + \frac{i}{3} f_{\mu\nu\rho} Tr[p_\mu, p_\nu] p_\rho \\
&\rightarrow 8N^2 \left[\frac{1}{2} \left(\frac{\beta}{f} \right)^4 - \frac{2}{3} \left(\frac{\beta}{f} \right)^3 \right] \frac{f^4}{2^5 n^{\frac{2}{3}} N^{\frac{1}{3}}} (r_1 + r_2 + r_3).
\end{aligned} \tag{3.3}$$

Here the large N limit is taken when we proceed from the 1st to 2nd line ($l_1, l_2, l_3 \gg 1$) and the following ratios which measure the relative sizes of S^2 's are introduced

$$r_1 = \left(\frac{l_1 l_1}{l_2 l_3} \right)^{\frac{2}{3}}, \quad r_2 = \left(\frac{l_2 l_2}{l_1 l_3} \right)^{\frac{2}{3}}, \quad r_3 = \left(\frac{l_3 l_3}{l_1 l_2} \right)^{\frac{2}{3}}. \tag{3.4}$$

The leading term of the one loop effective action in the large N limit can be evaluated as

$$\begin{aligned}
\Gamma_{1-loop} &= \frac{1}{2} Tr \log(P^2 \delta_{\mu\nu}) - Tr \log P^2 \\
&\rightarrow 8N^2 \left(\log \beta + \frac{1}{3} \log \frac{N}{8n} \right. \\
&\quad \left. + \frac{1}{128} \int_0^4 dX dY dZ \log(r_1 X + r_2 Y + r_3 Z) \right).
\end{aligned} \tag{3.5}$$

The magnitude of the two loop effective action can be estimated by the following power counting argument. The two loop amplitude diverges as the eighth power of the cutoff ($N^{\frac{8}{3}}$)

while it is suppressed by a power of N which is associated with the interaction vertices. Since the one loop amplitude is $O(N^2)$, we need to adopt the 't Hooft coupling ($N^{\frac{1}{3}}/f^4$) in such a way to make the tree contribution of $O(N^2)$ as well. With such a choice, the two loop amplitude scales as $N^{\frac{4}{3}}$. Since the tree and the one loop amplitude scale as N^2 , we can safely ignore the two loop amplitude in the large N limit. The analogous argument shows that we can ignore all higher loop contributions and the one loop effective action becomes exact in the large N limit.

In this way, we obtain the effective action in the large N limit as

$$\begin{aligned} \Gamma = & 8N^2 \left(\left[\frac{1}{2} \left(\frac{\beta}{f} \right)^4 - \frac{2}{3} \left(\frac{\beta}{f} \right)^3 \right] \frac{1}{2^5 \lambda_1^2} (r_1 + r_2 + r_3) \right. \\ & \left. + \log \beta + \frac{1}{3} \log \frac{N}{8n} + \frac{1}{128} \int_0^4 dX dY dZ \log(r_1 X + r_2 Y + r_3 Z) \right), \end{aligned} \quad (3.6)$$

where λ_1 is the 't Hooft coupling:

$$\lambda_1^2 = \frac{n^{\frac{2}{3}} N^{\frac{1}{3}}}{f^4}. \quad (3.7)$$

To minimize the effective action, we have to solve $\frac{\partial \Gamma}{\partial \beta} = 0$. This condition determines the scale factor β as

$$\frac{\beta}{f} = \frac{1}{4} + \frac{1}{2} \sqrt{\frac{1}{4} + g(A)} + \frac{1}{2} \sqrt{\frac{1}{2} - g(A) + \frac{1}{4\sqrt{\frac{1}{4} + g(A)}}}, \quad (3.8)$$

where

$$g(A) = \frac{2^{\frac{5}{3}} A}{3^{\frac{1}{3}} (9A + \sqrt{3\sqrt{27A^2 - 128A^3}})^{\frac{1}{3}}} + \frac{(9A + \sqrt{3\sqrt{27A^2 - 128A^3}})^{\frac{1}{3}}}{6^{\frac{2}{3}}}, \quad (3.9)$$

and $A = 2^5 \lambda_1^2$. We find that this solution exists in the weak coupling regime where

$$\lambda_1^2 = \frac{n^{\frac{2}{3}} N^{\frac{1}{3}}}{f^4} < \frac{3^3}{2^{12}} (r_1 + r_2 + r_3) \simeq 0.0198 \quad \text{at} \quad r_1 = r_2 = r_3 = 1. \quad (3.10)$$

Beyond this critical point, the fuzzy $S^2 \times S^2 \times S^2$ background no longer exists. Just like [9], this point separates the background distributions between the collapsed phase and the fuzzy $S^2 \times S^2 \times S^2$ phase.

After plugging (3.6) into (3.8), we can estimate the effective action by performing the triple integrals numerically for each λ_1 . Fig.1 shows $\Gamma/8N^2$ against $t = l_1/l_3 = l_2/l_3$ for $N = 8 \times 10^6$ and $\lambda_1^2 = 0.0197, 0.005, 0.001$. $t = 1$ corresponds to the most symmetric fuzzy

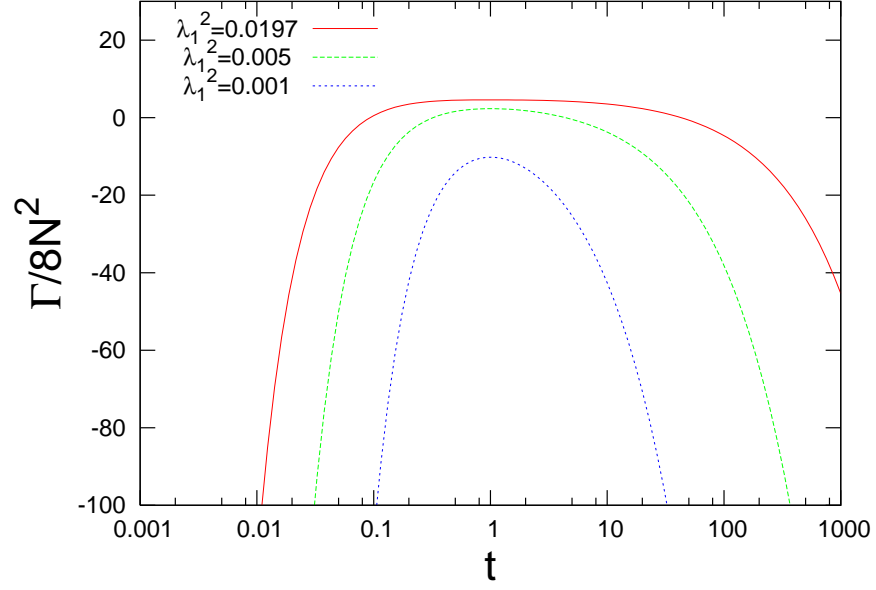


Figure 1: $\Gamma/8N^2$ against $t = l_1/l_3 = l_2/l_3$ for $N = 8 \times 10^6$ and $\lambda_1^2 = 0.0197, 0.005, 0.001$.

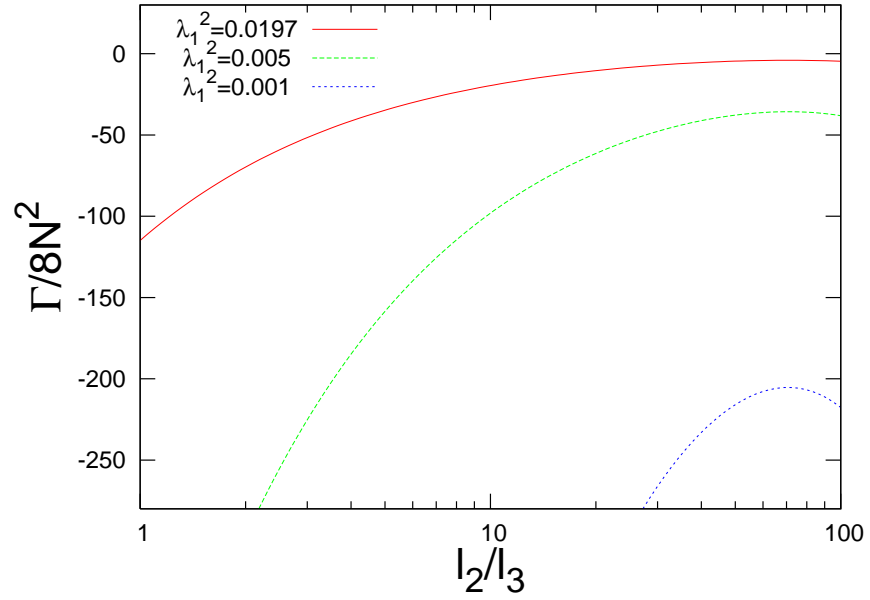


Figure 2: $\Gamma/8N^2$ against l_2/l_3 for $l_1/l_3 = 100$, $N = 8 \times 10^6$ and $\lambda_1^2 = 0.0197, 0.005, 0.001$.

$S^2 \times S^2 \times S^2$ background. As $t \rightarrow \infty$, the background approaches fuzzy $S^2 \times S^2$ while it approaches fuzzy S^2 as $t \rightarrow 0$.

From Fig.1 we can see that fuzzy $S^2 \times S^2 \times S^2$ background is not stable. To explore whether it tends to decay into $S^2 \times S^2$ or S^2 , let us examine the behavior of the effective action in more details. When we fix $l_1/l_3 = 100$ and vary l_2/l_3 between 1 to 100, the action behaves as Fig.2. Here, $l_2/l_3 = 100$ and $l_2/l_3 = 1$ in Fig.2 correspond to $t = 100$ and $t = 0.01$ in Fig.1 respectively because of the equivalence of the three S^2 's. As we observe Γ is smooth with respect to l_1/l_3 and l_2/l_3 , and does not develop a local minimum, we can convince ourselves that S^2 is favored in this example. The situation like this holds for the other combinations of l_1/l_3 and l_2/l_3 when one of them is large enough. Therefore we conclude that fuzzy $S^2 \times S^2 \times S^2$ background is not stable in the action (3.1), and it decays toward fuzzy S^2 .

4 Deformed IIB matrix model with Myers term

In this section, we study a deformed IIB matrix model with Myers term whose action is

$$S = -\frac{1}{4}Tr[A_\mu, A_\nu]^2 - \frac{1}{2}Tr\bar{\psi}\Gamma_\mu[A_\mu, \psi] + \frac{i}{3}f_{\mu\nu\rho}Tr[A_\mu, A_\nu]A_\rho. \quad (4.1)$$

We investigate the stability of the fuzzy $S^2 \times S^2 \times S^2$ background (2.5) which is a classical solution in this action ².

The effective action can be evaluated by extending the procedures in [7]. The results in the large N limit are

$$\begin{aligned} \Gamma_{tree} &\simeq -n^{\frac{2}{3}}N^{\frac{4}{3}}\frac{1}{24}\frac{f^4N^{\frac{1}{3}}}{n^{\frac{4}{3}}}(r_1 + r_2 + r_3), \\ \Gamma_{1-loop} &\simeq n^{\frac{2}{3}}N^{\frac{4}{3}}\frac{1}{8}\int_0^4 dX \int_0^4 dL \int_0^4 dA \frac{1}{r_1X + r_2L + r_3A}, \\ \Gamma_{2-loop} &\simeq -n^{\frac{2}{3}}N^{\frac{4}{3}}\frac{n^{\frac{4}{3}}}{f^4N^{\frac{1}{3}}}\left(\frac{3}{2}f_3 + 4f_4\right), \end{aligned} \quad (4.2)$$

f_3 and f_4 are defined as the following multiple integrals

$$\begin{aligned} f_3 &= \int_0^4 \frac{dXdYdZdLdMdNdAdBdC}{(r_1X + r_2L + r_3A)(r_1Y + r_2M + r_3B)(r_1Z + r_2N + r_3C)} \\ &\quad \times W(X, Y, Z)W(L, M, N)W(A, B, C), \\ f_4 &= \int_0^4 \frac{dXdYdZdLdMdNdAdBdC}{(r_1X + r_2L + r_3A)^2(r_1Y + r_2M + r_3B)^2(r_1Z + r_2N + r_3C)} \end{aligned}$$

²We need not generalize f into β here since there are no tadpoles at the one loop level .

$$\begin{aligned}
& \times \left[r_1^2 XY + r_2^2 LM + r_3^2 AB + \frac{1}{2r_3}(Z - X - Y)(N - L - M) \right. \\
& \times \left. + \frac{1}{2r_2}(Z - X - Y)(C - A - B) + \frac{1}{2r_1}(N - L - M)(C - A - B) \right] \\
& \times W(X, Y, Z)W(L, M, N)W(A, B, C),
\end{aligned} \tag{4.3}$$

where $W(X, Y, Z)$ is the asymptotic formula of Wigner's $6j$ symbols which appear in the interaction vertices:

$$\begin{aligned}
l_1^3 \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_1 & l_1 \end{matrix} \right\}^2 & \simeq W(j_1^2, j_2^2, j_3^2) \\
& = \frac{1}{2\pi \sqrt{\frac{YZ(4-Y)(4-Z)}{4} - \left(X - \frac{2Y+2Z-YZ}{2}\right)^2}}.
\end{aligned} \tag{4.4}$$

This approximation is valid in the uniformly large angular momentum regime. Since the effective action is highly divergent, we argue that this approximation is exact in the large N limit.

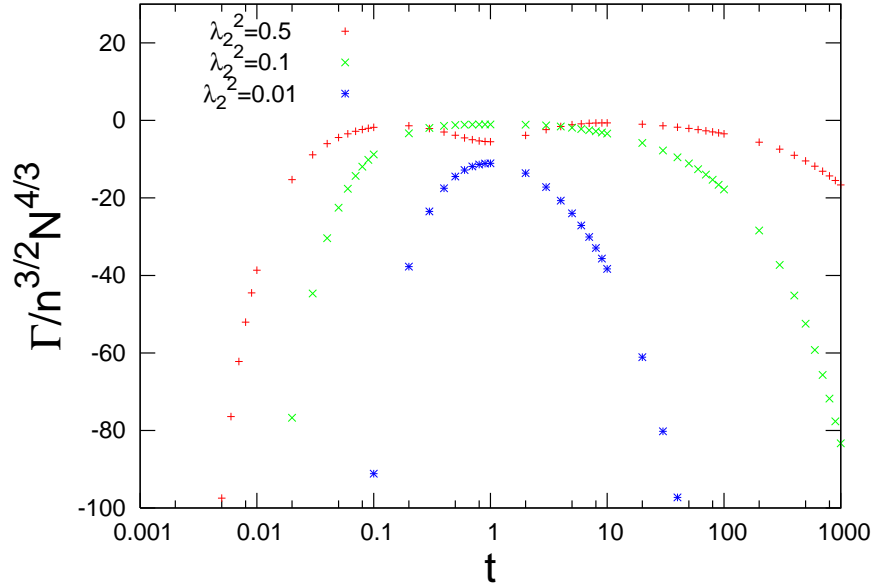


Figure 3: $\Gamma/n^{3/2}N^{4/3}$ against $t = l_1/l_3 = l_2/l_3$ for $\lambda_2^2 = 0.5, 0.1, 0.01$.

The effective action at the 2-loop level is

$$\Gamma = n^{\frac{2}{3}} N^{\frac{4}{3}} \left[-\frac{r_1 + r_2 + r_3}{24\lambda_2^2} + \frac{1}{8} \int_0^4 \frac{dX dL dA}{r_1 X + r_2 L + r_3 A} - \lambda_2^2 \left(\frac{3}{2} f_3 + 4 f_4 \right) \right]. \tag{4.5}$$

It is indeed of $O(N^{\frac{4}{3}})$ after choosing the 't Hooft coupling λ_2 :

$$\lambda_2^2 = \frac{n^{\frac{4}{3}}}{f^4 N^{\frac{1}{3}}}. \quad (4.6)$$

In the same way as in the previous section, we have estimated this effective action numerically. From Fig.3 we can observe again that the fuzzy $S^2 \times S^2 \times S^2$ background is not stable. The same analysis in the previous section can be used here to determine into which configuration it tends to decay. From such an analysis we conclude that the fuzzy $S^2 \times S^2 \times S^2$ background decays toward fuzzy S^2 . It is interesting to observe that $S^2 \times S^2 \times S^2$ becomes a local minimum of the effective action at $t = 1$ when the coupling λ_2 is strong enough.

5 IIB matrix model analysis with respect to spins

After investigating bosonic and supersymmetric models with Myers term, we evaluate the effective action for IIB matrix model (2.1) with the background (2.5). The results are ³

$$\begin{aligned} \Gamma_{tree} &\simeq n^{\frac{2}{3}} N^{\frac{4}{3}} \frac{1}{8} \frac{f^4 N^{\frac{1}{3}}}{n^{\frac{4}{3}}} (r_1 + r_2 + r_3), \\ \Gamma_{1-loop} &\simeq O\left(N^{\frac{2}{3}}\right), \\ \Gamma_{2-loop} &\simeq n^{\frac{2}{3}} N^{\frac{4}{3}} \frac{n^{\frac{4}{3}}}{f^4 N^{\frac{1}{3}}} \frac{1}{2} f_3. \end{aligned} \quad (5.1)$$

We can explicitly check that the effective action scales as $N^{\frac{4}{3}}$ at the two loop level with the following choice of the 't Hoot coupling

$$\begin{aligned} \Gamma &= n^{\frac{2}{3}} N^{\frac{4}{3}} \left(\frac{1}{8\lambda_2^2} (r_1 + r_2 + r_3) + \frac{\lambda_2^2}{2} f_3 \right), \\ \lambda_2^2 &= \frac{n^{\frac{4}{3}}}{f^4 N^{\frac{1}{3}}}. \end{aligned} \quad (5.2)$$

Since the 't Hoot coupling which is set by the overall scale of the background becomes a dynamical variable, we can minimize the effective action with respect to it for fixed representations:

$$\Gamma_{min} = 2\sqrt{\Gamma_{tree}\Gamma_{2-loop}}. \quad (5.3)$$

We can now explore the minimum of the effective action (5.3) with respect to the representations (l_1, l_2, l_3) . Fig.4 shows $\Gamma_{min}/n^{\frac{2}{3}} N^{\frac{4}{3}}$ against $t = l_1/l_3 = l_2/l_3$. We note that the

³The two loop amplitude Γ_{2-loop} is evaluated in the Appendix with generic scale factors.

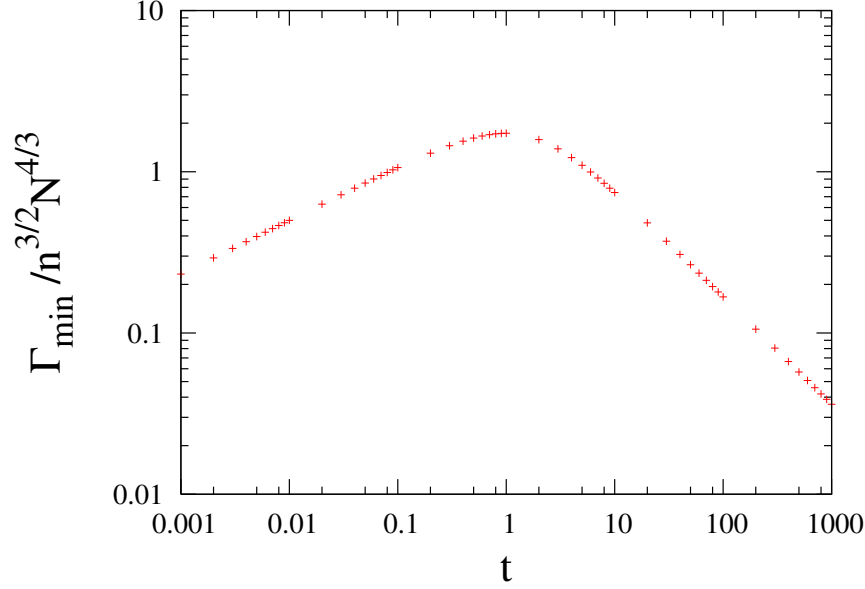


Figure 4: $\Gamma_{min}/n^{3/2}N^{4/3}$ against $t = l_1/l_3 = l_2/l_3$.

action is decreasing faster in the $t > 1$ region than $t < 1$ region. We may thus conclude that fuzzy $S^2 \times S^2 \times S^2$ is not stable and it tends to decay toward fuzzy $S^2 \times S^2$. This result is consistent with the previous investigations [7, 8].

We can further demonstrate that the action (5.1) reduces to that of $S^2 \times S^2$ when we take a limit $l_1, l_2 \gg l_3$. It is because we can reexpress (5.1) as

$$\begin{aligned}
\Gamma''_{tree} &= \frac{N}{8\lambda_3^2} (r + 1/r + R), \\
\Gamma''_{2-loop} &= 2N\lambda_3^2 l_3^2 \int_0^4 dXdYdZdLdMdNdAdBdC \\
&\quad \times \frac{W(X, Y, Z)W(L, M, N)W(A, B, C)}{(rX + r^{-1}L + RA)(rY + r^{-1}M + RB)(rZ + r^{-1}N + RC)}, \\
N &\simeq 2l_1 2l_2 2l_3, \quad \lambda_3^2 = \frac{1}{f^4 2l_1 2l_2}, \quad R = \frac{l_3^2}{l_1 l_2}, \quad r = \frac{l_1}{l_2}.
\end{aligned} \tag{5.4}$$

Here, we set $n = 1$ for simplicity.

Fig.5 shows $\Gamma''_{min}/N = 2\sqrt{\Gamma''_{tree} \cdot \Gamma''_{2-loop}}/N$ against $t = l_1/l_3 = l_2/l_3$ for $N = 8 \times 10^9$. It demonstrates that the effective action scales in a 4d fashion as the background approaches $S^2 \times S^2$.

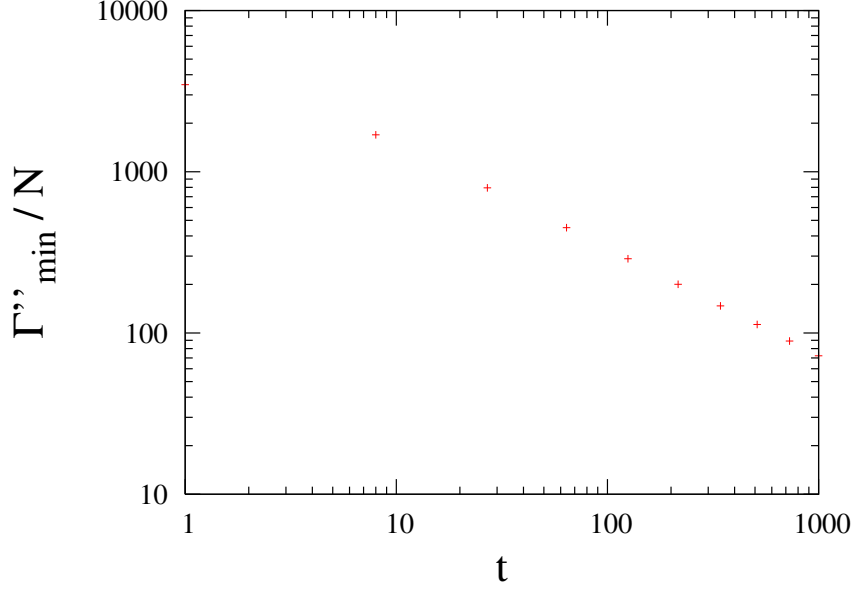


Figure 5: Γ''_{min}/N against $t = l_1/l_3 = l_2/l_3$.

6 Stability of fuzzy $S^2 \times S^2$ background

In this section, we investigate the stability of fuzzy $S^2 \times S^2$ background in detail with respect to the scale factors in addition to the change of the representations (spins). Our investigation is motivated by an analogous study for the fuzzy torus case [10].

We consider the background of the following type:

$$\begin{aligned}
p_\mu &= f_1 (\bar{j}_\mu \otimes 1) & (\mu = 1, 2, 3), \\
p_\mu &= f_2 (1 \otimes \hat{j}_\mu) & (\mu = 4, 5, 6), \\
p_\mu &= 0 & (\mu = 7, 8, 9, 0), \\
\chi &= 0.
\end{aligned} \tag{6.1}$$

Here we set $n = 1$ for simplicity. It generalizes the background in [8] where the identical scale factor $f_1 = f_2$ is assumed.

The effective actions are evaluated as ⁴

$$\Gamma'_{tree} = \frac{N'}{8\lambda'^2} \left(r q^2 + \frac{1}{r q^2} \right),$$

⁴It can again be read off from Γ_{2-loop} in the Appendix.

$$\begin{aligned}
\Gamma'_{1-loop} &= O(\log N'), \\
\Gamma'_{2-loop} &= 4N'\lambda'^2 \int_0^4 dXdYdZdLdMdN \\
&\quad \times \frac{(rq^2X + \frac{1}{rq^2}L)}{(rqX + \frac{1}{rq}L)^2(rqY + \frac{1}{rq}M)(rqZ + \frac{1}{rq}N)} \\
&\quad \times W(X, Y, Z)W(L, M, N),
\end{aligned} \tag{6.2}$$

where

$$N' = 2l_1 2l_2, \quad \lambda'^2 = \frac{1}{f_1^2 f_2^2 N'}, \quad q = \frac{f_1}{f_2}, \quad r = \frac{l_1}{l_2}. \tag{6.3}$$

Fig.6 and Fig.7 show $\Gamma'_{min}/N' = 2\sqrt{\Gamma'_{tree} \cdot \Gamma'_{2-loop}/N'}$ against r and q . The points represented by squares possess smaller effective actions than that of the most symmetric point $r = q = 1$. Furthermore \sqrt{rq} (the scale ratios of the two S^2) also decreases in this domain. Therefore we find that fuzzy $S^2 \times S^2$ background is not stable when the both spins and scale factors are allowed to change at the two loop level.

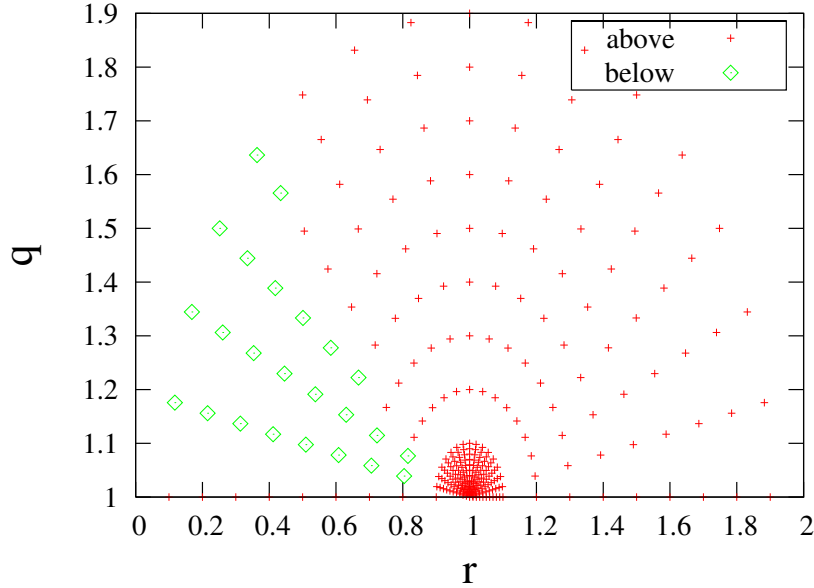


Figure 6: 3D plot: Γ'_{min}/N' against r and q . $r - q$ section.

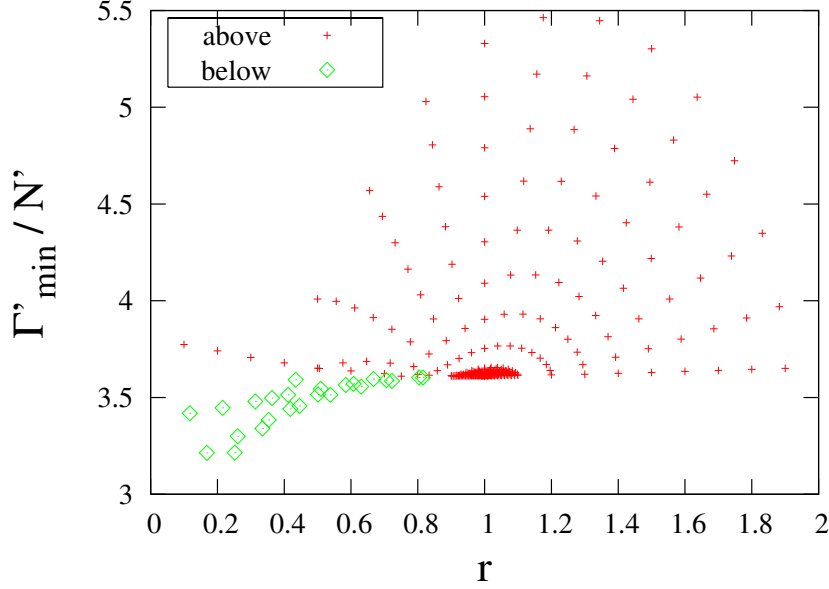


Figure 7: 3D plot: Γ'_{min}/N' against r and q . $r - \Gamma'_{min}/N'$ section.

7 Local stability of the backgrounds

In the previous sections, we have investigated the stability of the backgrounds globally by performing the multiple integrals numerically. In this section we also investigate the local stability of the most symmetric background with respect to the small variations of the spins and scale factors.

In $S^2 \times S^2$ case, the minimum of the effective action (6.2) under the variations of $r = 1 + \delta, q = 1 + \epsilon, (\delta, \epsilon \ll 1)$ becomes

$$\begin{aligned}
\frac{\Gamma'_{min}}{N'} &= \frac{2\sqrt{\Gamma'_{tree} \cdot \Gamma'_{2-loop}}}{N'} \equiv \frac{G}{4}, \\
G &= \left\langle \frac{(X + q^4 r^2 L)(1 + q^4 r^2) r^2}{(X + q^2 r^2 L)^2 (Y + q^2 r^2 M)(Z + q^2 r^2 N)} \right\rangle \\
&\simeq 2F' + \left(-\frac{7}{2}F' + 6|C'|^2 + 6|\varepsilon|^2\right)\delta^2 \\
&\quad + 2(|C'|^2 + |\varepsilon'|^2)\epsilon^2 + (-4F' + 8|C'|^2 + 8|\varepsilon'|^2)\delta\epsilon,
\end{aligned} \tag{7.1}$$

where

$$F' \equiv \left\langle \frac{1}{(X + L)(Y + M)(Z + N)} \right\rangle,$$

$$\begin{aligned}
|C'|^2 &\equiv \left\langle \frac{L^2}{(X+L)^3(Y+M)(Z+N)} \right\rangle, \\
|\varepsilon'|^2 &\equiv \left\langle \frac{LM}{(X+L)^2(Y+M)^2(Z+N)} \right\rangle, \\
\langle \dots \rangle &\equiv \int_0^4 dX dY dZ dL dM dN \dots W(X, Y, Z) W(L, M, N).
\end{aligned} \tag{7.2}$$

We can estimate $F', |C'|^2, |\varepsilon'|^2$ numerically as follows

$$\begin{aligned}
F' &= 3.263930 \pm 0.19 \times 10^{-4}, \\
|C'|^2 &= 1.056778 \pm 0.94 \times 10^{-5}, \\
|\varepsilon'|^2 &= 0.8630541 \pm 0.68 \times 10^{-5}.
\end{aligned} \tag{7.3}$$

Let us consider the following ratio:

$$\frac{G}{F'} = 2 + 0.0292\delta^2 + 1.18\epsilon^2 + 0.706\epsilon\delta. \tag{7.4}$$

To explore the stability of the background, we evaluate the eigenvalues of the quadratic forms in δ and ϵ above. The eigenvalues are

$$1.28, \quad -0.07. \tag{7.5}$$

Since we find a negative eigenvalue, we conclude that $S^2 \times S^2$ is not stable in such a direction around the most symmetric point.

The same analysis can be applied to $S^2 \times S^2 \times S^2$. In this case, a relevant background corresponding to (6.1) is

$$\begin{aligned}
p_\mu &= f_1(\bar{j}_\mu \otimes 1 \otimes 1) \otimes 1_n \quad (\mu = 1, 2, 3), \\
p_\mu &= f_2(1 \otimes \hat{j}_\mu \otimes 1) \otimes 1_n \quad (\mu = 4, 5, 6), \\
p_\mu &= f_3(1 \otimes 1 \otimes \tilde{j}_\mu) \otimes 1_n \quad (\mu = 7, 8, 9), \\
p_0 &= 0, \\
\chi &= 0.
\end{aligned} \tag{7.6}$$

We evaluate the two loop effective action for this generic background in the Appendix. The minimum of the effective action with respect to the 't Hooft coupling is

$$\Gamma_{min} = \frac{N^{4/3} H^{1/2}}{2^{1/2}},$$

$$\begin{aligned}
H &= \left\langle \frac{(X + q_2^4 r_2^2 L + q_3^4 r_3^2 A)(1 + q_2^4 r_2^2 + q_3^4 r_3^2)(r_2^2 r_3^2)^{\frac{2}{3}}}{(X + q_2^4 r_2^2 L + q_3^4 r_3^2 A)^2 (Y + q_2^2 r_2^2 M + q_3^2 r_3^2 B)(Z + q_2^2 r_2^2 N + q_3^2 r_3^2 C)} \right\rangle, \\
\langle \dots \rangle &\equiv \int_0^4 dX dY dZ dL dM dN dA dB dC \dots \\
&\quad \times W(X, Y, Z) W(L, M, N) W(A, B, C), \\
r_2 &= \frac{l_2}{l_1}, \quad r_3 = \frac{l_3}{l_1}, \quad q_2 = \frac{f_2}{f_1}, \quad q_3 = \frac{f_3}{f_1}.
\end{aligned} \tag{7.7}$$

The functional dependence of H on $r_i = 1 + \delta_i, q_i = 1 + \epsilon_i$ is

$$\begin{aligned}
H &= 3F + (9|C|^2 + 9|\varepsilon|^2 - \frac{8}{3}F)(\delta_2^2 + \delta_3^2) - (\frac{17}{3}F - 9|C|^2)\delta_2\delta_3 \\
&\quad + (9|C|^2 + 9|\varepsilon|^2 - \frac{3}{2}F)(\epsilon_2^2 + \epsilon_3^2) + (F - 3|C|^2)\epsilon_2\epsilon_3 \\
&\quad + (12|C|^2 + 12|\varepsilon|^2 - \frac{8}{3}F)(\delta_2\epsilon_2 + \delta_3\epsilon_3) \\
&\quad + (-\frac{3}{2}F - 6|C|^2 + 12|\varepsilon|^2)(\delta_2\epsilon_3 + \delta_3\epsilon_2),
\end{aligned} \tag{7.8}$$

where

$$\begin{aligned}
F &\equiv \left\langle \frac{1}{(X + L + A)(Y + M + B)(Z + N + C)} \right\rangle, \\
|C|^2 &\equiv \left\langle \frac{A^2}{(X + L + A)^3 (Y + M + B)(Z + N + C)} \right\rangle, \\
|\varepsilon|^2 &\equiv \left\langle \frac{AB}{(X + L + A)^2 (Y + M + B)^2 (Z + N + C)} \right\rangle.
\end{aligned} \tag{7.9}$$

We can estimated $F, |C|^2, |\varepsilon|^2$ numerically as follows

$$\begin{aligned}
F &= 3.993 \pm 0.25 \times 10^{-2}, \\
|C|^2 &= 0.6032 \pm 0.46 \times 10^{-3}, \\
|\varepsilon|^2 &= 0.4656 \pm 0.11 \times 10^{-2}.
\end{aligned} \tag{7.10}$$

It is again useful to consider the following ratio:

$$\begin{aligned}
\frac{H}{F} &= 3 - 0.26(\delta_2^2 + \delta_3^2) - 4.3\delta_2\delta_3 \\
&\quad + 0.91(\epsilon_2^2 + \epsilon_3^2) - 0.55\epsilon_2\epsilon_3 \\
&\quad + 0.55(\delta_2\epsilon_2 + \delta_3\epsilon_3) - 1.0(\delta_2\epsilon_3 + \delta_3\epsilon_2).
\end{aligned} \tag{7.11}$$

The eigenvalues of this quadratic form are

$$-2.43, \quad 2.39, \quad 0.69, \quad 0.65. \tag{7.12}$$

The existence of the negative eigenvalue implies the instability of this background. If we consider the δ_2, δ_3 subspace, the eigenvalues are

$$-2.41, \quad 1.89 \tag{7.13}$$

indicating the instability of this background in agreement with section 5.

8 Conclusions

In this paper we have investigated the effective action of matrix models with $S^2 \times S^2 \times S^2$ backgrounds at the two loop level. This class of 6 dimensional manifolds can be constructed by using $SU(2)$ algebra which facilitates us to evaluate the effective action. We can change the size of each S^2 by choosing different representations (spins) of $SU(2)$. Therefore we can probe manifolds of different dimensionality such as S^2 (2d), $S^2 \times S^2$ (4d) and $S^2 \times S^2 \times S^2$ (6d) by collapsing some of S^2 's. Since the background with the smallest effective action is most likely to be realized in a particular model, this investigation may shed light why our spacetime is 4 dimensional in IIB matrix model context.

In the previous investigations, the large N scaling behavior of the NC gauge theory on these manifolds has been clarified. In supersymmetric models, the effective action scales as N^2 , N and $N^{\frac{4}{3}}$ in 2, 4 and 6 dimensional manifolds respectively. It is always $O(N^2)$ and the one loop approximation is exact in bosonic models. We have indeed verified this scaling behavior for 6 dimensional spacetime at the two loop level. With the presence of Myers term, S^2 configuration is favored since the effective action can become negative. On the other hand, 4 dimensional spacetime is favored in IIB matrix model since the effective action is positive definite.

In accord with these expectations we find that the fuzzy $S^2 \times S^2 \times S^2$ background is not stable when we vary the ratios of the spins in the following matrix models

- 10d bosonic matrix model with Myers term,
- IIB matrix model with Myers term,
- IIB matrix model.

In the first two matrix models with Myers term, $S^2 \times S^2 \times S^2$ tends to decay toward S^2 . On the other hand $S^2 \times S^2 \times S^2$ tends to decay $S^2 \times S^2$ in the last one. These results are consistent with the previous works [7, 8].

We have further investigated the effective action around the symmetric $S^2 \times S^2$ in detail. Under the change of the ratios of the scale factors in addition to the spins, we find there are unstable directions of the $S^2 \times S^2$ background in IIB matrix model.

This instability does not imply that it eventually decays into S^2 since the effective action is $O(N^2)$ in such a limit. However this argument cannot be verified at the two loop level since the effective action for S^2 behaves as follows up to the two loop level

$$aN^3 f^4 + \frac{b}{f^4 N}, \quad (8.14)$$

where a, b are $O(1)$. It is therefore possible to make it $O(N)$ by choosing $1/f^4 N^2$ to be $O(1)$ which is the same order with 4d manifolds. However we argue that this is a two loop artifact since the n loop contribution can be estimated as $(1/f^4 N)^{n-1}$ and hence we need to adopt $1/f^4 N$ as the 't Hooft coupling. Nevertheless we cannot rule out the possible existence of instability of a simple product space of $S^2 \times S^2$. In this respect a more symmetric 4d spacetime such as $CP^2 = SU(3)/U(2)$ is interesting and it may not suffer from the instability found here since $SU(3)$ symmetry permits only the over all scale factor of the background [5, 14]. It is also likely that a manifold with higher symmetry may lower the effective action of IIB matrix model.

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Appendix A

In this appendix, we evaluate the two loop effective action of IIB matrix model in the $S^2 \times S^2 \times S^2$ background (7.6). Our calculation which is an extension of those in [7] incorporates the asymmetric scale factors. The result in the $S^2 \times S^2$ background (6.1) can be obtained by shrinking one of S^2 's.

We expand quantum fluctuations in terms of the tensor product of the matrix spherical harmonics:

$$a^\mu = \sum_{j m p q s t} a_{j m p q s t}^\mu (Y_{j m} \otimes Y_{p q} \otimes Y_{s t}),$$

$$\begin{aligned}
\varphi &= \sum_{j m p q s t} \varphi_{j m p q s t} (Y_{j m} \otimes Y_{p q} \otimes Y_{s t}), \\
b &= \sum_{j m p q s t} b_{j m p q s t} (Y_{j m} \otimes Y_{p q} \otimes Y_{s t}), \\
c &= \sum_{j m p q s t} c_{j m p q s t} (Y_{j m} \otimes Y_{p q} \otimes Y_{s t}).
\end{aligned} \tag{A.1}$$

Here the sums of j , p and s run up to $2l_1$, $2l_2$ and $2l_3$ respectively. Then the propagators are derived from the kinetic terms of (2.8):

$$\begin{aligned}
\langle a^\mu a^\nu \rangle &= \sum_{j m p q s t} \left(P^2 \delta_{\mu\nu} + 2i f_{\mu\nu\rho} P^\rho \right)^{-1} (Y_{j m} \otimes Y_{p q} \otimes Y_{s t}) (Y_{j m}^\dagger \otimes Y_{p q}^\dagger \otimes Y_{s t}^\dagger), \\
\langle \varphi \bar{\varphi} \rangle &= \sum_{j m p q s t} (-\Gamma_\mu P_\mu)^{-1} (Y_{j m} \otimes Y_{p q} \otimes Y_{s t}) (Y_{j m}^\dagger \otimes Y_{p q}^\dagger \otimes Y_{s t}^\dagger), \\
\langle cb \rangle &= \sum_{j m p q s t} \frac{1}{P^2} (Y_{j m} \otimes Y_{p q} \otimes Y_{s t}) (Y_{j m}^\dagger \otimes Y_{p q}^\dagger \otimes Y_{s t}^\dagger).
\end{aligned} \tag{A.2}$$

We exclude the singlet state $j = p = s = 0$ in the propagators. To calculate the leading contributions in the large N limit, we expand the boson and the fermion propagators as

$$\begin{aligned}
\left(P^2 \delta_{\mu\nu} + 2i f_{\mu\nu\rho} P^\rho \right)^{-1} &\simeq \frac{\delta_{\mu\nu}}{P^2} - 2i \frac{f_{\mu\nu\rho} P^\rho}{P^4} + 4 \frac{I_{\mu\nu}(P)}{P^6}, \\
(-\Gamma_\mu P_\mu)^{-1} &\simeq \frac{\Gamma^\mu P_\mu}{P^2} + \frac{i f_{\mu\nu\sigma} \Gamma^{\mu\nu\rho} P_\sigma P_\rho}{2 P^4} - \frac{\Gamma \cdot f^2 \cdot P}{P^4} \\
&\quad + \frac{P \cdot f^2 \cdot P P^\mu \Gamma_\mu}{P^6}.
\end{aligned} \tag{A.3}$$

We have introduced the following tensor

$$I_{\mu\nu} \equiv (\bar{\delta}_{\mu\nu} \bar{P}^2 - \bar{P}_{\mu\nu}) f_1^2 + (\hat{\delta}_{\mu\nu} \hat{P}^2 - \hat{P}_{\mu\nu}) f_2^2 + (\tilde{\delta}_{\mu\nu} \tilde{P}^2 - \tilde{P}_{\mu\nu}) f_3^2. \tag{A.4}$$

The symbols $\bar{}$, $\hat{}$ and $\tilde{}$ denote the sub-spaces whose Lorentz indices μ run over $(1, 2, 3)$, $(4, 5, 6)$ and $(7, 8, 9)$ respectively. We also introduce

$$P \cdot f^2 \cdot P \equiv f_1^2 \bar{P}^2 + f_2^2 \hat{P}^2 + f_3^2 \tilde{P}^2. \tag{A.5}$$

Using these propagators, we can calculate the contributions to the two loop effective action from various interaction vertices as follows.

4-gauge boson vertex is

$$V_4 = -\frac{1}{4} \text{Tr}[a_\mu, a_\nu]^2. \tag{A.6}$$

The leading contribution to the two loop effective action is

$$\langle -V_4 \rangle = \left\langle \frac{1}{P_1^2 P_2^2} \left\{ -45 + 6 \frac{P_3 \cdot f^2 \cdot P_3}{P_1^2 P_2^2} - 12 \frac{P_2 \cdot f^2 \cdot P_2}{P_1^2 P_2^2} - 72 \frac{P_1 \cdot f^2 \cdot P_1}{P_1^4} \right\} \right\rangle_P. \tag{A.7}$$

We introduce the wave functions and averages as

$$\begin{aligned}
\Psi_{123} &\equiv Tr(Y_{j_1 m_1} Y_{j_2 m_2} Y_{j_3 m_3}) Tr(Y_{p_1 q_1} Y_{p_2 q_2} Y_{p_3 q_3}) Tr(Y_{s_1 t_1} Y_{s_2 t_2} Y_{s_3 t_3}), \\
\langle X \rangle_P &\equiv \sum_{j_i, p_i, s_i, m_i, q_i, t_i} \Psi_{123}^* X \Psi_{123}, \\
P_i^\mu(Y_{j_i m_i} Y_{p_i q_i} Y_{s_i t_i}) &\equiv [p_\mu, Y_{j_i m_i} Y_{p_i q_i} Y_{s_i t_i}].
\end{aligned} \tag{A.8}$$

We define following functions:

$$\begin{aligned}
F_1 &= \left\langle \frac{1}{P_1^4 P_2^4} \right\rangle_P, \\
\tilde{g}_1 &= \left\langle \frac{P_2 \cdot f^2 \cdot P_2}{P_1^4 P_2^4} \right\rangle_P, \\
g_1 &= \left\langle \frac{P_1 \cdot f^2 \cdot P_1}{P_1^6 P_2^2} \right\rangle_P, \\
g_2 &= \left\langle \frac{P_3 \cdot f^2 \cdot P_3}{P_1^4 P_2^4} \right\rangle_P.
\end{aligned} \tag{A.9}$$

Then

$$\langle -V_4 \rangle = -45F_1 - 12\tilde{g}_1 - 72g_1 + 6g_2. \tag{A.10}$$

Ghost vertex is

$$V_g = Tr b [p_\mu, [a_\mu, c]]. \tag{A.11}$$

Their contribution is

$$\frac{1}{2} \langle V_g V_g \rangle = F_2 + 4H_2. \tag{A.12}$$

Here

$$\begin{aligned}
F_2 &= \left\langle \frac{P_2 \cdot P_3}{P_1^2 P_2^2 P_3^2} \right\rangle_P, \\
H_2 &= \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^6 P_2^2 P_3^2} \right\rangle_P,
\end{aligned} \tag{A.13}$$

and

$$P_i \cdot I(j) \cdot P_k \equiv P_i^\mu I_{\mu\nu}(P_j) P_k^\nu. \tag{A.14}$$

3-gauge boson vertex is

$$V_3 = -Tr P_\mu a_\nu [a_\mu, a_\nu]. \tag{A.15}$$

Their contribution is

$$\begin{aligned}
\frac{1}{2} \langle V_3 V_3 \rangle &= 9F_1 - 9F_2 + 12F_3 + 8g'_1 - 4\tilde{g}'_1 + 2g_2 \\
&\quad + 32H_1 - 36H_2 - 16H_3 + 12H_4 - 4H_5.
\end{aligned} \tag{A.16}$$

Newly introduced functions are defined as

$$\begin{aligned}
F_3 &= \left\langle \frac{P_1 \cdot f^2 \cdot P_1}{P_1^4 P_2^2 P_3^2} \right\rangle_P, \\
g'_1 &= g_1 - \frac{1}{N} \sum_{j,p,s} (2j+1)(2p+1)(2s+1) \\
&\quad \times \frac{f_1^4 j(j+1) + f_2^4 p(p+1) + f_3^4 s(s+1)}{[f_1^2 j(j+1) + f_2^2 p(p+1) + f_3^2 s(s+1)]^4}, \\
\tilde{g}'_1 &= \tilde{g}_1 - \frac{1}{N} \sum_{j,p,s} (2j+1)(2p+1)(2s+1) \\
&\quad \times \frac{f_1^4 j(j+1) + f_2^4 p(p+1) + f_3^4 s(s+1)}{[f_1^2 j(j+1) + f_2^2 p(p+1) + f_3^2 s(s+1)]^4}, \\
H_1 &= \left\langle \frac{P_1 \cdot I(2) \cdot P_1}{P_1^2 P_2^6 P_3^2} \right\rangle_P, \\
H_3 &= \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^4 P_2^4 P_3^2} \right\rangle_P, \\
H_4 &= \left\langle \frac{P_1 \cdot I(2) \cdot P_1}{P_1^4 P_2^4 P_3^2} \right\rangle_P, \\
H_5 &= \left\langle \frac{P_2 \cdot I(1) \cdot P_3}{P_1^2 P_2^4 P_3^4} \right\rangle_P.
\end{aligned} \tag{A.17}$$

Fermion vertex is

$$V_f = -\frac{1}{2} Tr \bar{\varphi} \Gamma_\mu [a_\mu, \varphi]. \tag{A.18}$$

Their contribution is

$$\begin{aligned}
\frac{1}{2} \langle V_f V_f \rangle &= -64F_2 + (-16\tilde{g}'_1 + 8g_2 + 16F_3 + 32H_4) \\
&\quad -32F_3 + 64g'_1 + 32\tilde{g}'_1 - 16g_2 + 64H_2 + 64H_3.
\end{aligned} \tag{A.19}$$

After summing up (A.10), (A.12), (A.16) and (A.19), we find the 2-loop effective action:

$$\begin{aligned}
\Gamma_{2-loop} &= 4F_3 + 32H_1 + 32H_2 + 48H_3 + (12 + 32)H_4 - 4H_5 \\
&= 4F_3.
\end{aligned} \tag{A.20}$$

It is because

$$\begin{aligned}
H_1 + H_2 &= 0, \\
H_3 + H_4 &= 0, \\
H_3 - H_5 &= 0.
\end{aligned} \tag{A.21}$$

The explicit form of F_3 is

$$\begin{aligned}
F_3 = & \sum_{j_i, p_i, s_i} \\
& \times \frac{(2j_1 + 1)(2p_1 + 1)(2s_1 + 1) [f_1^4 j_1(j_1 + 1) + f_2^4 p_1(p_1 + 1) + f_3^4 s_1(s_1 + 1)]}{[f_1^2 j_1(j_1 + 1) + f_2^2 p_1(p_1 + 1) + f_3^2 s_1(s_1 + 1)]^2} \\
& \times \frac{(2j_2 + 1)(2p_2 + 1)(2s_2 + 1)}{[f_1^2 j_2(j_2 + 1) + f_2^2 p_2(p_2 + 1) + f_3^2 s_2(s_2 + 1)]} \\
& \times \frac{(2j_3 + 1)(2p_3 + 1)(2s_3 + 1)}{[f_1^2 j_3(j_3 + 1) + f_2^2 p_3(p_3 + 1) + f_3^2 s_3(s_3 + 1)]} \\
& \times \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_1 & l_1 \end{matrix} \right\}^2 \left\{ \begin{matrix} p_1 & p_2 & p_3 \\ l_2 & l_2 & l_2 \end{matrix} \right\}^2 \left\{ \begin{matrix} s_1 & s_2 & s_3 \\ l_3 & l_3 & l_3 \end{matrix} \right\}^2.
\end{aligned} \tag{A.22}$$

In the large N limit, we can use the following approximations

$$\begin{aligned}
j_i(j_i + 1) & \rightarrow j_i^2, \\
2j_i + 1 & \rightarrow 2j_i, \\
\sum_{j_i=1}^{2l_i} & \rightarrow \int_0^{2l_i} dj_i, \\
l_1^3 \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_1 & l_1 \end{matrix} \right\}^2 & \rightarrow W(j_1^2, j_2^2, j_3^2).
\end{aligned} \tag{A.23}$$

We also define new variables as

$$\begin{aligned}
X &= j_1^2, \quad Y = j_2^2, \quad Z = j_3^2, \\
L &= p_1^2, \quad M = p_2^2, \quad N = p_3^2, \\
A &= s_1^2, \quad B = s_2^2, \quad C = s_3^2.
\end{aligned} \tag{A.24}$$

Finally F_3 assumes the following expression

$$\begin{aligned}
F_3 = & \frac{l_1 l_2 l_3}{(f_1 f_2 f_3)^{4/3}} \int_0^4 dX dY dZ dL dM dN dA dB dC \\
& \times \frac{q_1^2 r_1 X + q_2^2 r_2 L + q_3^2 r_3 A}{(q_1 r_1 X + q_2 r_2 L + q_3 r_3 A)^2} \\
& \times \frac{1}{(q_1 r_1 Y + q_2 r_2 M + q_3 r_3 B)(q_1 r_1 Z + q_2 r_2 N + q_3 r_3 C)} \\
& \times W(X, Y, Z) W(L, M, N) W(A, B, C),
\end{aligned} \tag{A.25}$$

where

$$\begin{aligned}
r_1 &= \left(\frac{l_1 l_1}{l_2 l_3} \right)^{\frac{2}{3}}, \quad r_2 = \left(\frac{l_2 l_2}{l_1 l_3} \right)^{\frac{2}{3}}, \quad r_3 = \left(\frac{l_3 l_3}{l_1 l_2} \right)^{\frac{2}{3}}, \\
q_1 &= \left(\frac{f_1 f_1}{f_2 f_3} \right)^{\frac{2}{3}}, \quad q_2 = \left(\frac{f_2 f_2}{f_1 f_3} \right)^{\frac{2}{3}}, \quad q_3 = \left(\frac{f_3 f_3}{f_1 f_2} \right)^{\frac{2}{3}}.
\end{aligned} \tag{A.26}$$

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